GROWTH TIGHTNESS OF GROUPS WITH NONTRIVIAL FLOYD BOUNDARY

WEN-YUAN YANG

Abstract. We show that groups with nontrivial Floyd boundary are growth tight. As a consequence, all non-elementary relatively hyperbolic groups are growth tight.

1. Introduction

Let G be a finitely generated group with a finite generating set S. Denote by B_n the number of all elements in G with word length less than n. The *growth rate* $\delta_{G,S}$ of G relative to S is defined as

$$\delta_{G,S} = \lim_{n\to\infty} n^{-1} \ln B_n,$$

where the limit exists by the sub-additive inequality.

The notion of growth tightness of groups was first introduced by R. Grigorchuk and P. de la Harpe in [9].

Definition 1.1. The group G is *growth tight* if for every finite generating set S, it is true that $\delta_{G,S} > \delta_{G/\Gamma,\bar{S}}$ for any infinite normal subgroup $\Gamma < G$, with $\bar{S}$ denoting the canonical image of S in G/Γ.

It is an interesting question to ask which classes of groups are growth tight. In [1], Arzhantseva-Lysenok showed that hyperbolic groups are growth tight. In another direction, Sambusetti [12] proved that free products of two groups are growth tight, which was used to construct the first examples of groups whose minimal growth rate is not realized by any generating set. This answered an open question in [9].

In the present paper, the growth tightness of groups is established for a more general class of groups which admit a non-trivial Floyd boundary. The notion of Floyd boundary was introduced in [4] to obtain a nontrivial compactification of one-ended groups. Given a finite generating set S, the construction of Floyd boundary really depends on the choice of the *Floyd function* $f : \mathbb{N} \to \mathbb{R}_{>0}$ which satisfies the *summable* condition

$$\sum_{n=1}^{\infty} f(n) < \infty$$

and the λ-*delay* property

$$\lambda < f(n+1)/f(n) < 1$$

for $\lambda \in]0,1[$. Let's denote the Floyd boundary by $\partial_{S,f}G$ with reference to these parameters. The Floyd boundary $\partial_{S,f}G$ is *nontrivial* if it consists of at least three

Date: Jan 20, 2013.
2000 *Mathematics Subject Classification.* Primary 20F65, 20F67.
This research is supported by the ERC starting grant GA 257110 "RaWG".

points. By construction, the particular shape of Floyd boundary depends crucially on f and S. However, we observe that the non-triviality of Floyd boundary is rather a group property in a precise sense (cf. Lemma 2.6). See section 2 for the detail and more references.

Our main result can be stated in the following way.

Theorem 1.2. *If G has a nontrivial Floyd boundary $\partial_{S,f}G$ for some finite generating set S and Floyd function f, then G is growth tight.*

To explain consequences of Theorem 1.2, we examine two particular types of Floyd functions which have been appeared in the literatures.

In [4], Floyd considered the case of "polynomial type", for instance $f(n) = 1/(n^2+1)$. In this case, the shapes of Floyd boundary behave nicely in the spirit of coarse geometry: the construction using different finite generating sets shall lead to the same homeomorphic Floyd boundaries.

The exponential type of Floyd function $f(n) = \lambda^n$ for $\lambda \in]0,1[$ has also been exploited by Gerasimov-Potyagailo [5], [8],[6] and [7] in the study of relatively hyperbolic groups. In [5], Gerasimov showed that for a non-elementary relatively hyperbolic group G with a finite generating set S, there exists $\lambda \in]0,1[$ such that there is a continuous surjective map from the Floyd boundary $\partial_{S,f}G$ for $f(n) = \lambda^n$ to the Bowditch boundary of G. Hence all non-elementary relatively hyperbolic groups have nontrivial Floyd boundary. As a consequence, we obtain the following corollary, extending the results of Arzhantseva-Lysenok [1] and Sambusetti [12].

Corollary 1.3. *Non-elementary relatively hyperbolic groups are growth tight.*

Finally let's make some remarks about the relation between relatively hyperbolic groups and groups with nontrivial Floyd boundaries. As aforementioned, every non-elementary relatively hyperbolic groups have nontrivial Floyd boundary for some Floyd functions. It was asked by Olshanskii-Osin-Sapir in [11] whether its converse is true. It is worth to point out that they formulated the question for the type of Floyd function $f(n) = 1/(n^2+1)$.

In [13], we showed that their question is equivalent to the following question concerning about peripheral structures: whether every group hyperbolic relative to a collection of Non-Relatively Hyperbolic subgroups acts geometrically finitely on its Floyd boundary. Here the term "non-relatively hyperbolic" group means that the group is not hyperbolic relative to any collection of proper subgroups.

Hence it is not yet known whether the groups with nontrivial Floyd boundary coincide with the class of (non-elementary) relatively hyperbolic groups. The techniques in our paper only exploit the geometry of groups imposed by the non-triviality of Floyd boundary.

The structure of this paper is as follows. In section 2, we recall necessary details about the Floyd boundary. A large portion is borrowed from [14] to discuss the notion of contracting sets in groups with nontrivial Floyd boundary. In section 3, we discuss a critical gap criterion of Dalbo-Peigne-Picaud-Sambusetti [3]. In section 4, these ingredients are brought together to prove Theorem 1.2.

2. Preliminaries

2.1. **Notations and Conventions.** Let (Y,d) be a geodesic metric space. Given a subset X and a number $U \geq 0$, let $N_U(X) = \{y \in Y : d(y,X) \leq U\}$. Denote by $\|X\|$ the diameter of X with respect to d.

Given a point $y \in Y$ and subset $X \subset Y$, let $\Pi_X(y)$ be the set of points x in X such that $d(y,x) = d(y,X)$. Define the projection of a subset A to X as $\Pi_X(A) = \cup_{a\in A}\Pi_X(a)$.

Let p be a path in Y with initial and terminal endpoints p_- and p_+ respectively. Denote by $\ell(p)$ the length of p. Given two points $x, y \in p$, denote by $[x,y]_p$ the subpath of p going from x to y.

A path p going from p_- to p_+ induces a first-last order as we describe now. Given a property (P), a point z on p is called the *first point* satisfying (P) if z is among the points w on p with the property (P) such that $\ell([p_-,w]_p)$ is minimal. The *last point* satisfying (P) is defined in a similarly way.

2.2. **Contracting subsets and admissible paths.** We start with a general discussion about contracting subsets in a geodesic metric space (Y,d). Most results in this section are simplified versions of those in Section 2 in [14].

Definition 2.1 (Contracting subset)**.** Given $\mu \geq 0, \epsilon > 0$, a subset X is called (μ,ϵ)-*contracting* in Y if the following inequality holds

$$\|\Pi_X(q)\| < \epsilon,$$

for any geodesic q with $d(q,X) \geq \mu$. A collection of (μ,ϵ)-contracting subsets is referred to as a (μ,ϵ)-*contracting system.*

The notion of contracting sets exhibits a kind of hyperbolic feature in a more general context. For instance, the following lemma says that a geodesic triangle is thin with respect to contracting sets.

Lemma 2.2. [14] *Let X be a (μ,ϵ)-contracting set and $x \in X$ and $y \notin X$ two points. Then there is a constant $\sigma > 0$ such that for any projection point o of y to X, we have $d(o,[x,y]) \leq \sigma$.*

Given a function $\nu : \mathbb{R} \to \mathbb{R}_+$, two subsets $X, X' \subset Y$ have ν-*bounded intersection* if $\|N_U(X) \cap N_U(X')\| < \nu(U)$ for any $U \geq 0$. As usual, a collection $\mathbb{X}$ of subsets have ν-*bounded intersection* if any two subsets in $\mathbb{X}$ do so.

Lemma 2.3. [14] *Let X, X' be two (μ,ϵ)-contracting subsets with ν-bounded intersection in Y. Then they have B-bounded projection for some $B > 0$:*

$$\|\Pi_X(X')\| < B, \ \|\Pi_{X'}(X)\| < B.$$

Recall that a geodesic p is said to be τ-*orthogonal* to a subset $X \subset Y$ if $\|\Pi_X(p)\| \leq \tau$ for some constant $\tau > 0$.

Let $\mathbb{X}$ be a (μ,ϵ)-contracting system with ν-bounded intersection, and fix $\tau > 0$. We define the notion of an admissible path.

Definition 2.4 (Admissible Paths)**.** Given $D \geq 0$, a D-*admissible path* γ is a concatenation of geodesics in Y such that the following conditions hold:

(1) Exactly one geodesic p_i of any two consecutive ones in γ has two endpoints in a contracting subset $X_i \in \mathbb{X}$,
(2) Each p_i has length bigger then D, except that p_i is the first or last geodesic in γ, and
(3) For each X_i, the geodesics with one endpoint in X_i are τ-orthogonal to X_i.

The collection $\{X_i\}$ are referred to as *associated contracting subsets* of the admissible path γ.

For definiteness in the sequel, usually write $\gamma = p_0 q_1 p_1 \dots q_n p_n$ and assume that p_i has endpoints in a contracting subset $X_i \in \mathbb{X}$ and the following conditions hold.

(1) $\ell(p_i) > D$ for $0 < i < n$.
(2) $q_i \in \mathcal{L}$ is τ-orthogonal to both X_{i-1} and X_i for $1 \le i \le n$.

In [14], we proved that D-admissible paths are quasigeodesics for sufficiently large D. However, this result will not be used in this paper. We only need the following lemma, which can be proven as Lemma 2.21 in [14].

Lemma 2.5. *There are constants $D = D(\epsilon, \mu, \nu, \tau) > 0, R = R(\epsilon, \mu, \nu, \tau) > 0$ such that the following statement holds.*

Let γ be a D_0-admissible path for $D_0 > D$ and X_i an associated contracting subset of γ. Assume that α is a geodesic between γ_-, γ_+. Then there exist points $z, w \in \alpha \cap N_\mu(X_i)$ such that $d(z, (p_i)_-) \le R,\ d(w, (p_i)_+) \le R$.

2.3. **Floyd boundary and hyperbolic elements.** We first recall the construction of Floyd boundary of a finitely generated group G. Let $f : \mathbb{N} \to \mathbb{R}_+$ be a summable function with λ-delay property as in the introduction.

Let S be a generating system and $\mathscr{G}(G, S)$ the Cayley graph of G with respect to S. Denote by d_S the combinatorial (word) metric on $\mathscr{G}(G, S)$. The *Floyd length* of each edge e in $\mathscr{G}(G, S)$ is set to be $f(n)$, where $n = d_S(1, e)$. This naturally defines a length metric ρ on $\mathscr{G}(G, S)$. Let $\overline{G}_{S,f}$ be the Cauchy completion of G with respect to ρ. The complement $\partial_{S,f} G$ of G in $\overline{G}_{S,f}$ is called its *Floyd boundary*. The Floyd boundary $\partial_{S,f} G$ of G is *nontrivial* if $\partial_{S,f} G$ contains at least three points. We refer the reader to [4], [10] and [8] for more detail.

By construction, the particular shape of Floyd boundary depends on the choice of f and the generating set S. However, the following lemma roughly says that the non-triviality of Floyd boundary is a group property.

Lemma 2.6. *Suppose that G has a nontrivial Floyd boundary $\partial_{S,f} G$ for some finite generating set S and Floyd function f. Then for any finite generating set T, there is a Floyd function g such that $\partial_{T,g} G$ is nontrivial.*

Proof. It is well-known that the identity map extends to a (K, K)-quasi-isometric map $\varphi : \mathscr{G}(G, T) \to \mathscr{G}(G, S)$. Suppose that $\partial_{S,f} G$ is nontrivial and the Floyd function f satisfies the λ-delay property for $\lambda \in]0, 1[$. We want to find a Floyd function g such that φ extends to a continuous map $\hat{\varphi} : \partial_{T,g} G \to \partial_{S,f} G$. This shows the conclusion.

Recall a result of Gerasimov-Potyagailo that if $f(n)/g(Kn)$ is upper bounded, then ϕ extends to a continuous map $\hat{\varphi} : \partial_{T,g} G \to \partial_{S,f} G$. Without loss of generality, we assume that $K = 2$. The general case follows from a finite number of repeated applications of the case $K = 2$.

Define $g(2n) = f(n)$ and $g(2n-1) = (f(n-1) + f(n))/2$, where $g(0) = 0$. It is obvious that $f(n)/g(2n) = 1$. It suffices to verify that $g(n)$ is a Floyd function. Direct computations show that g satisfies the $2\lambda/(1+\lambda)$-delay property. The proof is complete. □

In [10], Karlsson showed that the left multiplication on G extends to a convergence group action of G on $\partial_{S,f} G$. Then the elements in G can be classified into there classes by their fixed points(cf. [2]). In particular, an infinite order element h is called *hyperbolic* if it has exactly two fixed points h_-, h_+ in $\overline{G}_{S,f}$.

Let h be a hyperbolic element in G. The set $C(h_-, h_+)$ is defined to be the union of all geodesics in $\mathscr{G}(G, S)$ between h_- and h_+. Denote by $E(h)$ the stabilizer in G of the set $\{h_-, h_+\}$. In the following lemma, we collect the necessary facts which will used later.

Lemma 2.7. *Let h be a hyperbolic element in a finitely generated group G with non-trivial Floyd boundary. Then the following statements are true.*

1. *The group $E(h)$ acts cocompactly on $C(h_-, h_+)$ and contains $\langle h \rangle$ as finite index subgroup.*
2. *If $gC(h_-, h_+) = g'C(h_-, h_+)$ for $g, g' \in G$, then $gE(h) = g'E(h)$.*
3. *The set $C(h_-, h_+)$ is (μ, ϵ)-contracting for some constants $\mu \geq 0, \epsilon > 0$.*
4. *The collection $\mathbb{X} = \{gC(h_-, h_+) : g \in G\}$ have ν-bounded intersection for some function $\nu : \mathbb{R} \to \mathbb{R}$.*

Proof. These statements stems from the well-known fact that $E(h)$ or $\langle h \rangle$ acts cocompactly on $\overline{G}_{S,f} \setminus \{h_-, h_+\}$. Then the statements (1) and (3) are consequences of Propositions 4.2.2 and 8.2.4 in [8] respectively. The statement (2) is obvious by definition of $E(h)$. The statement (4) can be proven by the same proof of Proposition 5.1.4 in [8]. $\square$

3. A critical gap criterion

The aim of this section is to recall a critical gap criterion due to Dalbo-Peigne-Picaud-Sambusetti. Their criterion is adapted to our specific context and will be used in the proof of Theorem 1.2.

3.1. **Poincaré series.** Recall that G is a finitely generated group with a finite generating set S. Denote by b_n the number of elements in G with word length n. Define $B_n = \sum_{i=0}^{n} b_i$. Consider the Poincaré series of G,

$$\Theta_{G,d_S}(s) = \sum_{g \in G} \exp(-s \cdot d_S(1, g)), s > 0.$$

Observe that $\Theta_{G,d_S}(s)$ has the same convergence as the series:

$$\Theta'_{G,d_S}(s) = \sum_{n=0}^{\infty} B_n \exp(-sn), s > 0,$$

as we have $\Theta'_{G,d_S}(s)(1 - \exp(-s)) = \Theta_{G,d_S}(s)$.

The *critical exponent* of the series $\Theta_{G,d_S}(s)$ (or $\Theta_{G,d_S}(s)$) is the limit superior $\limsup_{n\to\infty} n^{-1} \ln B_n$, which turns out to be the growth rate $\delta_{G,S}$ of G relative to S. This is because that $B_{n+m} \leq B_n B_m$ for any $n, m \in \mathbb{N}$. Note that the series $\Theta_{G,d_S}(s)$ is convergent for $t > \delta_{G,S}$ and divergent for $t < \delta_{G,S}$.

In fact, the Poincaré series is divergent at the critical exponent.

Lemma 3.1. *The series $\Theta_{G,d_S}(s)$ is divergent at $s = \delta_{G,S}$.*

Proof. Since $\ln B_n$ satisfies the sub-additive inequality, it follows by Fekete Lemma that

$$\lim_{n\to\infty} n^{-1} \ln B_n = \inf_{n \geq 0} n^{-1} \ln B_n.$$

Hence we have $B_n \geq \exp(n\delta_{G,S})$ for any $n \geq 0$. Therefore,

$$\Theta'_{G,d_S}(s) = \sum_{n=0}^{\infty} B_n \exp(-sn) \geq \sum_{n=0}^{\infty} \exp(n(\delta_{G,S} - s)) \geq \frac{1}{1-\exp(\delta_{G,S}-s)}.$$

Let $s \to \delta_{G,S}$. Then $\Theta_{G,d_S}(s)$ is divergent at $s = \delta_{G,S}$. $\square$

Let A be any subset in G. The *Poincaré series associated to* A can be defined in the same way:

$$\Theta_{A,d_S}(s) = \sum_{a\in A} \exp(-sd_S(1,a)),$$

with the *critical exponent* defined as follows

$$\delta_{A,S} = \limsup_{r\to\infty} r^{-1} \ln B_r,$$

where B_r is the number of elements in G with word length less then $r (r \geq 1)$.

Given a normal subgroup Γ in G, let $\bar{G} = G/\Gamma$ be the quotient, and $\pi : G \to \bar{G}$ the epimorphism. Denote by $\bar{S} = \pi(S)$ the natural image of S and by $\bar{d}$ the induced word metric on $\bar{G}$.

We first record the following simple lemma, saying that $\bar{d}$ is the same as the quotient metric induced by the left action of Γ on G.

Lemma 3.2. *For any $\bar{g} \in \bar{G}$, we have $\bar{d}(1, \bar{g}) = d(1, \pi^{-1}(\bar{g}))$.*

Proof. Let $\bar{d}(1, \bar{g}) = l$ and $g\Gamma = \pi^{-1}(\bar{g})$, where $g \in G$. Then $\bar{g} = s_1 s_2 \cdots s_l \Gamma$, where $s_i \in S$. Hence $d(1, g\Gamma) \leq l$.

Suppose to the contrary that $d(1, g\Gamma) < l$. Then there exists $\gamma \in \Gamma$ such that $d(1, g\gamma) < l$. Hence $g\gamma = s_1 s_s \cdots s_m$, where $s_i \in S$ and $m < l$. Then $\bar{g} = s_1 s_s \cdots s_m \Gamma$. This contradicts that $\bar{d}(1, \bar{g}) = l$. Therefore we obtain that $\bar{d}(1, \bar{g}) = d(1, \pi^{-1}(\bar{g}))$. $\square$

A section map $\iota : \bar{G} \to G$ is defined as follows. For each $\bar{g} \in \bar{G}$, choose a representative element $g \in \pi^{-1}(\bar{g})$ such that $d(g, 1) = d(g, \Gamma)$, since $g\Gamma = \Gamma g$. Let $\iota(\bar{g}) = g$.

Moreover, we can assume that $\iota(\bar{g}^{-1}) = \iota(\bar{g})^{-1}$. In fact, let $g \in \pi^{-1}(\bar{g})$ such that $d(g, 1) = d(g, \Gamma)$. By Lemma 3.2, we have $d(1, g^{-1}\Gamma) = d(1, g\Gamma)$. Hence $d(g^{-1}, 1) = d(g^{-1}, \Gamma)$.

Set $G_\Gamma = \iota(\bar{G})$. We see the relation

$$\Theta_{\bar{G},\bar{d}_S}(s) = \Theta_{G_\Gamma,d_S}(s).$$

Moreover, we have the following.

Lemma 3.3. *Suppose G is a finitely generated group with a generating system S. Let A be a subset and $R > 0$ a constant such that $N_R(A) = G$. Then the series $\Theta_{A,d_S}(s)$ has the same convergence as $\Theta_{G,d_S}(s)$. In particular, $\delta_{G,S} = \delta_{A,S}$.*

Proof. It suffices to note the following inequality

$$\sum_{a\in A} \exp(-sd(1,a)) \leq \sum_{g\in G} \exp(-sd(1,g)) \leq M \exp(sR) \sum_{a\in A} \exp(-sd(1,a)),$$

where M is the cardinality of the ball of radius R. $\square$

3.2. **A critical gap criterion.** Let A be a subset in G and h an element in G. Denote by $\mathcal{W}(A,h)$ the set of all words W in the alphabet $A \cup \{h\}$ such that exactly one of two consecutive letters in W is either h or in the set A. Let

$$\kappa : \mathcal{W}(A,h) \to G$$

be the evaluation map and its image in G denoted by $A \star h$.

In [3], Dalbo-Peigne-Picaud-Sambusetti introduced the following critical gap criterion by constructing the free product sets. Since it plays an important role in proving Theorem 1.2 and its proof is rather short, we include their proof for the completeness.

Lemma 3.4. [3] *Let A be a subset in G and h be a nontrivial element in $G \setminus A$. Assume that $\kappa : \mathcal{W}(A,h) \to G$ is injective. If the series $\Theta_{A,d_S}(s)$ is divergent at $s = \delta_{A,S}$, then $\delta_{A\star h,S} > \delta_{A,S}$.*

Proof. We estimate the Poincaré series of $A \star h$ as follows:

$$\begin{aligned}
\sum_{g\in A\star h} \exp(-s\cdot d_S(1,g)) \geq & \sum_{n=1}^{\infty} \sum_{a_1,\ldots,a_n\in A} \exp(-s\cdot d_S(1,a_1h\cdots a_nh)) \\
\geq & \sum_{n=1}^{\infty} \sum_{a_1,\ldots,a_n\in A} \exp(-s\cdot d_S(1,a_1h_1))\cdots \exp(-s\cdot d_S(1,a_nh_n)) \\
\geq & \sum_{n=1}^{\infty} (\sum_{a\in A} \exp(-s\cdot d_S(1,ah)))^n.
\end{aligned}$$

Since it is assumed that $\sum_{a\in A} \exp(-s\cdot d_S(1,ah)) = \infty$ for $s = \delta_A$. Then there exists some $s > \delta_{A,S}$ such that $\sum_{a\in A} \exp(-s\cdot d_S(1,ah)) > 1$. Hence we have $\delta_{A\star h,S} > \delta_{A,S}$. □

4. Proof of Theorem 1.2

4.1. **Constructing free subsets in G.** Fix a generating set S of G and denote by d the word metric relative to S. Let Γ be a normal subgroup in G. Construct a symmetric set $G_\Gamma \leq G$ consisting of minimal representatives in $\bar{G} = G/\Gamma$ as in Section 3.

Let h be a hyperbolic element in Γ. In what follows, fix a basepoint o in $C(h_-,h_+)$.

Lemma 4.1. *There exists a constant $\tau = \tau(h) > 0$ such that the following properties hold:*

Given $g \in G_\Gamma$, any geodesic between o and go has bounded projection τ to the set $C(h_-,h_+)$.

Proof. By Lemma 2.7, the subgroup $\langle h\rangle$ acts cocompactly on $C(h_-,h_+)$. Let M be the diameter of the quotient $C(h_-,h_+)/\langle h\rangle$.

Let $g \in G_\Gamma$ and take any projection point z of go to $C(h_-,h_+)$. Then there exists an integer $m > 0$ such that $d(h^m(o),z) \leq M$. We shall show that $d(o,z) \leq 3M+2\delta$.

By Lemma 2.2, there is a constant $\sigma > 0$ such that $d(z,[o,go]) < \sigma$. Let w be the point on $[o,go]$ such that $d(z,w) < \sigma$. Hence we have $d(h^m(o),w) \leq M+\sigma$.

Observe that $d(o,w) \leq d(w,h^m(o))$. Indeed, suppose to the contrary that $d(o,w) > d(w,h^m(o))$. Then $d(go,o) = d(go,w)+d(w,o) > d(go,h^m(o))$. This gives a contradiction with the choice of $g \in \Gamma$ for which $d(o,go) = d(\Gamma o,go)$.

Hence $d(o, z) \le d(o, w) + d(w, h^m(o)) + d(h^m(o), z) \le 3M + 2\delta$. This shows the bounded projection of $[o, go]$ to $C(h_-, h_+)$. □

Given $A \subset G_\Gamma$ and $n > 0$, let $W = a_1 h^n a_2 h^n \dots a_k h^n$ be a word in $\mathcal{W}(A, h^n)$. We associate to W *a normal path* γ in $\mathscr{G}(G, S)$, which is a concatenation of geodesics $\gamma = p_1 q_1 \dots p_k q_k$ such that $\gamma_- = o$ and p_i, q_i are geodesics in $\mathscr{G}(G, S)$ which are labeled by a_i, h^n respectively.

Recall that there are $\mu \ge 0, \epsilon > 0$ such that $C(h_-, h_+)$ is a (μ, ϵ)-contracting set by Lemma 2.7. Moreover the collection $\mathbb{X} = \{gC(h_-, h_+) : g \in G\}$ is a (μ, ϵ)-contracting system with ν-bounded intersection for some function $\nu : \mathbb{R} \to \mathbb{R}$. These facts imply that the normal paths are admissible.

Lemma 4.2. *Given $A \subset G_\Gamma, n > 0$, let $W = a_1 h^n a_2 h^n \dots a_k h^n$ be a word in $\mathcal{W}(A, h^n)$, and $\gamma = p_1 q_1 \dots p_k q_k$ the normal path of W in $\mathscr{G}(G, S)$. Then γ is a D-admissible path, where $D = d(1, h^n)$.*

Proof. Observe that each geodesic q_i has both endpoints in a contracting subset

$$X_i = a_1 h^n a_2 h^n \dots a_i C(h_-, h_+) \in \mathbb{X}.$$

Now it suffices to see that each $p_i (1 < i < k)$ is τ-orthogonal to X_i for some $\tau > 0$. The case that p_i is τ-orthogonal to X_{i-1}(if defined) is similar. Translate X_i to $C(h_-, h_+)$ by $a_i^{-1} h^{-n} a_{i-1}^{-1} h^{-n} \dots a_1^{-1}$. Correspondingly, p_i is translated to a geodesic between $(a_i)^{-1} o$ and o. By the construction of G_Γ, we also have $(a_i)^{-1} \in G_\Gamma$. Then the conclusion follows from Lemma 4.1. □

Moreover, we have the following lemma by Lemma 2.5.

Lemma 4.3. *There exist constants $R = R(h) > 0$ and $N = N(h) > 0$ such that the following property holds.*

Given $A \subset G_\Gamma, n > N$, let $W = a_1 h^n a_2 h^n \dots a_k h^n$ be a word in $\mathcal{W}(A, h^n)$ and $\gamma = p_1 q_1 \dots p_k q_k$ the normal path of W. Assume that α is a geodesic between γ_-, γ_+. Then given an associated contracting subset X_i of γ, there exist $z, w \in \alpha \cap N_\mu(X_i)$ such that $d(z, (p_i)_-) \le R,\ d(w, (p_i)_+) \le R$.

The last step in proving Theorem 1.2 is that $\kappa : \mathcal{W}(A, h^n) \to G$ is injective for large n.

Lemma 4.4. *There exists $N = N(h) > 0$ such that the following property holds.*

Let $W = a_1 h^n a_2 \dots a_k h^n, W' = a_1' h^n a_2' \dots a_l' h^n$ be two words in $\mathcal{W}(A, h^n)$ for $n > N$ such that $\kappa(W) = \kappa(W')$. Then there is a constant $L = L(h, n) > 0$ we have $d(a_1 \Gamma, a_1' \Gamma) < L$.

Remark 4.5. As shown in [14], a long admissible path is a quasigeodesic. Hence so is the normal path of a word in $\mathcal{W}(A, h^n)$. However, the non-triviality of Lemma 4.4 comes from the fact that G_Γ is not a subgroup in G. Hence the cancellation in $W^{-1} W'$ would not necessarily give a word in $\mathcal{W}(A, h^n)$.

Proof. Let $\gamma = p_1 q_1 \dots p_k q_k, \gamma' = p_1' q_1' \dots p_l' q_l'$ be the normal paths for W, W' respectively. Take a geodesic α between o and go.

Note that h is a hyperbolic element in Γ. Then it follows from Lemma 2.7 that there is a constant $M > 0$ such that $E(h) \subset N_M(\Gamma)$ and $C(h_-, h_+) \subset N_M(\Gamma)$.

Consider contracting sets $X = a_1 C(h_-, h_+)$ and $X' = a_1' C(h_-, h_+)$. Without loss of generality, suppose that $X \ne X'$. Otherwise we see that $a_1^{-1} a_1' \in E(h)$ by

Lemma 2.7. It follows that $a_1^{-1}a_1' \in N_M(\Gamma)$ and then $d(a_1\Gamma, a_1'\Gamma) < M$. Setting $L = M$ suffices to finish the proof. Hence we assume that $X \neq X'$.

By Lemma 2.3, there is a constant $B > 0$ such that $\|\Pi_X(X')\| \leq B$. Let $R = R(h), N = N(h)$ be the constants given by Lemma 4.3. Choose N further such that the following inequality holds:

$$d(1, h^n) > 2R + 2M + B + \tau + 10\mu + 3\epsilon, \tag{1}$$

for any $n > N$.

Let z, w be the first and last points on γ respectively such that $z, w \in N_\mu(X) \cap \alpha$. By Lemma 4.3, we have that $d(z, (p_1)_-) \leq R,\ d(w, (p_1)_+) \leq R$. Similarly denote by u, v the first and last points on γ such that $u, v \in N_\mu(X') \cap \alpha$ and $d(u, (p_1')_-) \leq R,\ d(v, (p_1')_+) \leq R$.

We have the following two cases to analyze:

Case 1). The u(or v) lies in the segment $[z, w]_\gamma$. Assume that u is in $[z, w]_\gamma$. The other case is similar and simpler.

We see that $d(u, w) < \nu(\mu)$, as $u, w \in N_\mu(X) \cap N_\mu(X')$. Hence we obtain that $d(a_1 o, a_1' o) < d(h^n o, o) + R + \nu(\mu)$. It suffices to set

$$L = d(h^n o, o) + R + \nu(\mu).$$

Case 2). Segments $[z, w]_\gamma$ and $[u, v]_\gamma$ are disjoint. Without loss of generality, we can assume that $u, v \in [w, go]_\gamma$. The other case that $z, w \in [v, go]_\gamma$ is symmetric. We shall lead to a contradiction.

Without loss of generality, assume that the geodesic p_1' intersects nontrivially in the μ-neighborhood of X. The case that $p_1' \cap N_\mu(X) = \emptyset$ can be seen as a degenerated case and can be dealt with in a similar way. See Figure 1.

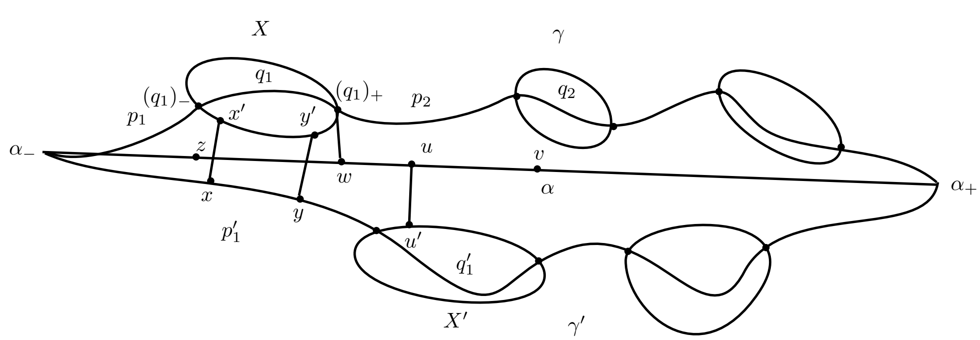

FIGURE 1. Proof of Lemma 4.4

Let x, y be the first and last points of p_1' such that $x, y \in N_\mu(X)$. We claim that $d(x, y) \leq 2\mu + 2M$. In fact, as $a_1' \in G_\Gamma$, we have $d(a_1' o, \Gamma o) = d(a_1' o, o)$. It follows that for any $x, y \in [o, a_1' o]$, we have $d(x\Gamma, y\Gamma) = d(x, y)$. Since $x, y \in N_\mu(X)$, we see that $d(x, a_1\Gamma) \leq \mu, d(y, a_1\Gamma) \leq \mu + M$. This implies that $d(x\Gamma, y\Gamma) \leq 2\mu + 2M$ as Γ is normal. Hence it is shown that $d(x, y) \leq 2\mu + 2M$.

Denote by x', y' projection points of x, y to X respectively. It follows that

$$d(x', y') \leq 4\mu + 2M. \tag{2}$$

Using projection, we estimate the following distance:

$$d(x', (q_1)_-) \le \|\Pi_X(p_1)\| + \|\Pi_X([o, x]_{p'_1})\| \le \tau + \epsilon. \tag{3}$$

where it follows from Lemma 4.1 and the fact that $[o, x]_{p'_1}$ lies outside $N_\mu(X)$.

Let u' be a projection point of u to X'. We estimate again by projection:

$$\begin{array}{rl} d(y', (q_1)_+) & \le \|\Pi_X([y, (p'_1)_+])\| + \|\Pi_X(X')\| + \|\Pi_X([u, u'])\| \\ & +\|\Pi_X([w, u]_\gamma)\| + \|\Pi_X([w, (q_1)_+])\| \\ & \le 2\epsilon + B + 2(R + 2\mu + \epsilon) \le 2R + B + 6\mu + 2\epsilon. \end{array} \tag{4}$$

Here we used the fact that the projection of a geodesic segment of length R to X is upper bounded by $R + 2\mu + \epsilon$. Then it follows that $\|\Pi_X([u, u'])\| \le R + 2\mu + \epsilon$, $\|\Pi_X([w, (q_1)_+])\| \le R + 2\mu + \epsilon$ in (4).

With (2), (3) and (4), we obtain that

$$\ell(q_1) < d(x', (q_1)_-) + d(x', y') + d(y', (q_1)_+) \le 2R + 2M + B + \tau + 10\mu + 3\epsilon.$$

This gives a contradiction with the choice of N in (1). Hence the case 2) is impossible. $\square$

4.2. **Proof of Theorem 1.2.** Let S be a finite generating system of G and $\Gamma \lhd G$. Construct, as in Section 3, a symmetric set $G_\Gamma \le G$ which consists of minimal representatives for $\bar{G} = G/\Gamma$. By construction we have $\Theta_{G,d_S}(s) = \Theta_{\bar{G},\bar{d}_S}(s)$.

Fix a hyperbolic element h in Γ. Let $n > N$ be a number and $L = L(h, n)$, where $N = N(h), L = L(h, n)$ are the constants given by Lemma 4.4.

We take a L-net A in G_Γ such that $d(a\Gamma, a'\Gamma) > L$ for any distinct $a, a' \in A$ and for any $g \in G$, there exists $a \in A$ such that $d(g\Gamma, a\Gamma) < L$. It follows from Lemma 4.4 that the evaluation map $\kappa : \mathcal{W}(A, h^n) \to G$ is injective.

By Lemma 3.3, it follows that $\delta_{A,S} = \delta_{G_\Gamma,S}$. Moreover, the Poincaré series $\Theta_{A,d_S}(s)$ has the same convergence as $\Theta_{\bar{G},\bar{d}_S}(s)$. Hence we see that $\Theta_{A,d_S}(s)$ is divergent at $s = \delta_{A,S}$ by Lemma 3.1.

By Lemma 3.4, one sees that $\delta_{G,S} \ge \delta_{A\star h^n,S} > \delta_{A,S}$. Therefore, it is shown that $\delta_{G,S} > \delta_{\bar{G},\bar{S}}$.

Acknowledgment. The author is grateful to Anna Erschler for helpful discussions and insightful thoughts. The author would like to thank Leonid Potyagailo for his constant encouragement and support over years.

References

1. G. Arzhantseva and I. Lysenok, *Growth tightness for word hyperbolic groups*, Math. Z. **241** (2002), no. 3, 597–611.
2. B. Bowditch, *Convergence groups and configuration spaces*, Geometric Group Theory Down Under (J. Cossey, C.F. Miller, W.D. Neumann, M. Shapiro, eds.), pp. 23–54, de Gruyter, 1999.
3. F. Dal'bo, M. Peigné, J.C. Picaud, and A. Sambusetti, *On the growth of quotients of kleinian groups*, Ergodic Theory and Dynamical Systems **31** (2011), no. 3, 835–851.
4. W. Floyd, *Group completions and limit sets of kleinian groups*, Inventiones Math. **57** (1980), 205–218.
5. V. Gerasimov, *Floyd maps for relatively hyperbolic groups*, Geom. Funct. Anal. (2012), no. 22, 1361 – 1399.
6. V. Gerasimov and L. Potyagailo, *Quasi-isometries and floyd boundaries of relatively hyperbolic groups*, arXiv:0908.0705, to appear in Journal of EMS, 2009.
7. ———, *Non-finitely generated relatively hyperbolic groups and floyd quasiconvexity*, arXiv:1008.3470, to appear in Groups, Geometry and Dynamics, 2010.
8. ———, *Quasiconvexity in the relatively hyperbolic groups*, arXiv:1103.1211, 2011.

9. R. Grigorchuk and P. de la Harpe., *On problems related to growth, entropy and spectrum in group theory*, J. Dyn. Control Syst. **3** (1997), no. 1, 51 – 89.
10. A. Karlsson, *Free subgroups of groups with non-trivial floyd boundary*, Comm. Algebra. **31** (2003), 5361–5376.
11. A. Olshanskii, D. Osin, and M. Sapir, *Lacunary hyperbolic groups*, Geom. Topol. **13** (2009), no. 4, 2051–2140, With an appendix by Michael Kapovich and Bruce Kleiner.
12. A. Sambusetti, *Growth tightness of free and amalgamated products*, Ann. Sci. École Norm. Sup. **série 35** (2002), no. 4, 477 – 488.
13. W. Yang, *Peripheral structures of relatively hyperbolic groups*, to appear in Journal für reine und angewandte Mathematik.
14. ———, *Combination of fully quasiconvex subgroups and its applications*, arXiv:1205.2994, 2012.

Le Dpartement de Mathmatiques de la Facult des Sciences d'Orsay, Universit Paris-Sud 11, France
E-mail address: yabziz@gmail.com